negatives of the Moon by MM. Lœwy and Puiseux, presented by Dr. Weinek; photographs of the lunar eclipse 1896 February 28, presented by G. J. Newbegin.

On the Systematic Errors of Measures on Photographic Plates.

By H. H. Turner, M.A., B.Sc., Savilian Professor.

(1) In the January number of the *Monthly Notices* the Astronomer Royal and Mr. Dyson gave an account of the work done at the Royal Observatory, Greenwich, in measuring and reducing a number of plates for the Astrographic Catalogue. The material thus available for the study of the accidental and systematic errors of such measures is most valuable, for the number of overlapping plates discussed brings out the power of the photographic method, and also its limitations, in a very clear manner.

(2) The present note is concerned with one section of the paper only, viz., that headed "Systematic Error in the Determination of a." It is therein shown that when a S.E. corner of one plate is compared with the N.W. corner of an overlapping plate, the deduced value of the constant a is positive; whereas when a S.W. corner is compared with a N.E. corner, the value of a is negative—a being one of the constants in a pair of linear equations

$$x_2 - x_1 = ax_1 + by_1 + c$$
$$y_2 - y_1 = dx_1 + ey_1 + f$$

which represent the differences between the coordinates on the two plates. Such a systematic error is cumulative, and prevents the stepping from one plate to another with anything like accuracy.

(3) The sources of such an error may be numerous—errors in the réseau or the measuring scale, optical distortion, &c. It is remarked in the paper that as yet no investigation of them has been made. The object of the present note is to consider one possible cause—viz. tilt of the plate. The error noted is small, though it becomes serious by accumulation; and it is possible that a slight want of perpendicularity of the plate to the line joining the centre of the object-glass and the centre of the plate might explain it in a manner developed in the sequel. I know that particular care has been taken to have this adjustment made at Greenwich, but the adjustments of a telescope cannot be examined every day, and possibly this particular adjustment may have been accidentally disturbed at some time. It is in any case advisable to consider the effect of such a disturbance.

(4) It has been shown in previous papers that if the normal

to the plate from the centre of the object-glass cuts the plate in the point (k, l) and not at the centre (0, 0), then the measured coordinates (x, y) will be related to the standard coordinates (ξ, η) in the following manner :

$$x=\frac{(1+a)\xi+b\eta+c}{1+k\xi+l\eta} \qquad y=\frac{d\xi+(1+e)\eta+f}{1+k\xi+l\eta}.$$

Where a, b, c, d, e, f, are all small, we have approximately

$$x=\xi-k\xi^2-l\xi\eta$$
$$y=\eta-k\xi\eta-l\eta^2.$$

(5) Suppose then that we first calculate ξ and η for the known stars from their R.A.s and N.P.D.s, as given by meridian observations, and then compare these standard coordinates with the measured coordinates. If in the first place we assume $k=0$, $l=0$ (*i.e.* that the plate is adjusted for tilt), we solve equations of the form

$$x-\xi=ax+by+c$$
$$y-\eta=dx+ey+f$$

and hence determine a, b, c, d, e, f. But if this assumption of no tilt is false, these deduced values of a, b, c, d, e, f, will be slightly erroneous. The errors will depend upon the positions of the known stars, and we must make some assumption on this point. The best is that the known stars are distributed uniformly over the plate, which is a square of side $4s$ say (where $s=30'$ approximately). Further, it is here assumed that the procedure adopted is to group all the stars in the N. half of the plate, all in the S. half, all in the E. half, and all in the W. half, as mentioned in previous papers.

(6) Now in considering the effect of tilt we may neglect all other errors; and since we may put $x=\xi$, $y=\eta$ in the small terms, we may consider therefore the equations

$$ax+by+c=-kx^2-lxy$$
$$dx+ey+f=-kxy-ly^2.$$

Integrating these over the N. half of the plate ($x=-2s$ to $x=+2s$, $y=0$ to $+2s$) we get

$$0\,.\,a+8s^3b+8s^2c\,.=\frac{32s^4}{3}k-0\,.\,l$$

$$0\,.\,d+8s^3e+8s^2f=-0\,.\,k-\frac{32s^4}{3}l.$$

From the S. half we should get

$$0\,.\,a-8s^3b+8s^2c=-\frac{32s^4}{3}k-0\,.\,l$$

$$0\,.\,d-8s^3e+8s^2f=-0\,.\,k\,.\,-\frac{32}{3}s^4l.$$

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Subtracting S. half from N. we get

$$b=0, \ e=0, \ c=-\frac{4s^2}{3}k, \ f=-\frac{4s^2}{3}l.$$

Similarly from the E. and W. halves of the plate we should get

$$a=0, \ d=0, \ c=-\frac{4s^2}{3}k, \ f=-\frac{4s^2}{3}l.$$

The measured coordinates

$$\xi-k\xi^2-l\xi\eta, \quad \eta-k\xi\eta-l\eta^2$$

are thus reduced, after comparison with ξ and η, to

$$\xi-k\xi^2-l\xi\eta+\frac{4s^2}{3}k, \quad \eta-k\xi\eta-l\eta^2+\frac{4s^2}{3}l.$$

(7) Now let another plate have its centre at $(+2s, +2s)$, and suppose its scale the same as that of the former and its orientation correct, and that the theoretical expressions have been applied to reduce the measures upon it to accordance with the measures on the first plate. If there were no error of tilt the coordinates of all stars would now exactly agree, except for accidental errors. But if the agate points against which the plates are brought into position for perpendicularity to the axis of the telescope be not correctly adjusted, there will be a systematic error of "tilt"—the same for both plates. From what precedes it will be seen that the ξ-coordinate of a star on the first plate (A) will now be

$$\xi-k\xi^2-l\xi\eta+\frac{4s^2}{3}k$$

and of the same star on the second plate (B)

$$\xi-k(\xi-2s)^2-l(\xi-2s)(\eta-2s)+\frac{4s^2}{3}k$$

and the residual difference will be

$$\xi_B-\xi_A=s\xi(4k+2l)+2s\eta l-4ls^2.$$

Similarly

$$\eta_B-\eta_A=s\eta(4l+2k)+2sk\xi-4ks^2.$$

Let us now consider plates overlapping the other corners viz.

Plate C centre $(-2s, +2s)$
,, D ,, $(-2s, -2s)$
,, E ,, $(+2s, -2s)$.

The corresponding equations can be easily deduced, and the whole set will be as follows :—

$$\xi_B - \xi_A = s\xi(+4k+2l) + 2s\eta l - 4ls^2$$
$$\xi_C - \xi_A = s\xi(-4k+2l) - 2s\eta l + 4ls^2$$
$$\xi_D - \xi_A = s\xi(-4k-2l) - 2s\eta l - 4ls^2$$
$$\xi_E - \xi_A = s\xi(+4k-2l) + 2s\eta l + 4ls^2$$

$$\eta_B - \eta_A = s\eta(+4l+2k) + 2s\xi k - 4ks^2$$
$$\eta_C - \eta_A = s\eta(+4l-2k) + 2s\xi k + 4ks^2$$
$$\eta_D - \eta_A = s\eta(-4l-2k) - 2s\xi k - 4ks^2$$
$$\eta_E - \eta_A = s\eta(-4l+2k) - 2s\xi k + 4ks^2.$$

(8) To estimate the magnitude of these terms it should be remarked that s is approximately $30'$, or ·0087 in circular measure ; and if $k = 10'$ (*i.e.* if the normal from the centre of the object-glass on to the plate cuts it only 2 réseau intervals away from the adopted centre) we have

$$4ks = \cdot 0087 \times \cdot 0087 \times \frac{4}{3} = \cdot 00010\,;$$

so that the differences of coordinates on two plates will be represented by terms such as

$$\xi_B - \xi_A = \cdot 00010\xi$$
$$\eta_B - \eta_A = \cdot 00005\xi + \cdot 00005\eta - \cdot 00010s.$$

These coefficients are just of the order of magnitude noticed by the Astronomer Royal and Mr. Dyson in their paper (*Monthly Notices*, lvi. pp. 125, 126).

The case of transformation from S.E. to N.W. corresponds to our transformation from C to A (or A to E) ; the case of transformation from S.W. to N.E. is that from B to A (or A to D). And we have in the first case

$$a_1 = s(-4k+2l),\ e_1 = s(4l-2k),\ (b+d)_1 = 2s(k-l)$$

and in the second case

$$a_2 = s(4k+2l),\ e_2 = s(4l+2k).\ (b+d)_2 = 2s(k+l).$$

Taking the means of the values given on p. 125 we have

$$a_1 = +\cdot 00011,\ e_1 = +\cdot 00003,\ (b+d)_1 = -\cdot 00003$$
$$a_2 = -\cdot 00012,\ e_2 = +\cdot 00001,\ (b+d)_2 = \cdot 00000$$

and we have thus six equations to determine k and l. Solving by least squares we get

$$2ks = -\cdot 000044,\ 2ls = +\cdot 000009$$

which give the values for a_1, a_2, &c., shown in column C below :

	O.	C.	O–C.
a_1	+ ·00011	+ ·00010	+ ·00001
e_1	+ ·00003	+ ·00006	– ·00003
$(b+d)_1$	– ·00003	– ·00005	+ ·00002
a_2	– ·00012	– ·00008	– ·00004
e_2	+ ·00001	– ·00003	+ ·00004
$(b+d)_2$	·00000	– ·00004	+ ·00004

the sums of the squares of the quantities being reduced in the proportion 284 to 62.

(9) The table which follows on p. 126 of the above paper gives the results when an overlapping plate is used to connect two plates in the same zone ; thus :

$$\begin{aligned} \xi_C - \xi_B &= (\xi_C - \xi_A) - (\xi_B - \xi_A) \\ &= -8ks\xi - 2ls\eta + 8ls^2 \end{aligned}$$

from above ; and

$$\eta_C - \eta_B = 0\xi - 4ks\eta + 8ks^2.$$

With the above values of k and l we should have for the expression of these residuals

$$a = -8ks = +\cdot00018 \quad b = -2ls = -\cdot00001$$
$$d = 0 \qquad\qquad e = -4ks = +\cdot00009.$$

The mean value of a given in the paper from seventeen plates is $+\cdot00021$. The values of e are not given in the paper, but were kindly furnished by the Astronomer Royal, on application, for all plates but one pair (2136–2227). Omitting this from a also we find mean values

$$a = \cdot00020, \quad e = -\cdot00004$$

and the observed and calculated values of e thus differ by ·00013. This certainly throws some doubt on the reality of the "tilt" as a cause of the errors under discussion, unless there is some numerical mistake. But there is no doubt of the importance of this adjustment for "tilt," if plates are to be connected with one another in this way ; and it appears from what precedes that such an error should, if sensible, be detected in the comparison of plates with overlapping plates by means of the criterion

$$a = 2e.$$

(10) In conclusion, I venture to make a simple suggestion. Why should each region be photographed on a different plate? It would strengthen the determination of systematic errors immensely if several regions were photographed on the same plate without disturbing the telescope very seriously, or the plate in its holder at all. The stars of different regions might be identified by making the displacements from the 6^m to the 3^m exposure in different directions, or of different magnitudes, and the cases where stars of one region interfered with those of another would be rare. To have more stars on one plate would make it easier to measure, and there would be economy in many ways—of time in changing plates and developing, and of expense in actual plates and reproduction if any. We are trying this method at Oxford to see how it works.

Note on Professor Turner's Paper on the Systematic Errors of Measures of Photographic Plates. By W. H. M. Christie, M.A., F.R.S., and F. W. Dyson, M.A.

Professor Turner points out that the systematic difference in the value of the constant a on the two halves of the photographic plates taken at the Royal Observatory, referred to in a paper in the *Monthly Notices* for January, may be due to a tilt of the

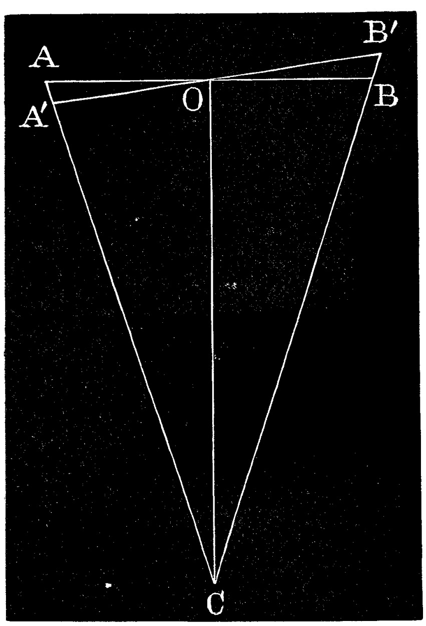

plate. It is easily seen geometrically that a tilt would have this effect. In the diagram O is the centre of a plate, C the centre